THE PENETRATION OF A JET INTO A CHANNEL

F. S. Vladimirov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 3, pp. 36–40, 1967

The method of S. A. Chaplygin [1], as generalized by S. V. Fal'kovich [2] to the case of a few characteristic velocities, is used to solve the two-dimensional problem of the penetration of a subsonic jet of compressible fluid flowing at an angle from a slit into a stream of the same fluid bounded by parallel walls. The problem is solved for the case of an incompressible fluid by passing to the asymptotic limit. Using the tables of [3] the compression coefficient is calculated for a stream of gas merged with an incompressible fluid.

§ 1. Let a two-dimensional, steady-state, adiabatic stream of gas with a density ρ_2 and a subsonic velocity v_2 move from left to right along a channel with parallel walls MON and ABCE (Fig. 1) and pass to infinity. This will be referred to as the main stream. A jet of the same gas flows out of a flat slit of width h with rectilinear parallel walls GB and FC, set at an angle λ to the channel wall ABCE. This gas has a density ρ_1 and subsonic velocity v_1 deep within the slit. It is assumed that after these streams meet, the gas jet breaks away from the channel wall at the point C and penetrates into the main stream, forming a discontinuity surface CD, which separates the combined stream from the space filled with gas which is at rest. We shall confine ourselves to the case when the boundary between the leading edge of the jet BK and the main stream is not a discontinuity line, but a stream line, common to the flows and having a continuous change of velocity along the boundary.

Let ρ_3 and v_3 be the gas density and velocity, respectively, at the surface of the jet CD, and δ be the width of the combined stream at infinity to the right. The coordinate origin is situated on the channel wall MON, and the x axis is in the direction of the flow, while the y axis passes through the point B. The coordinates of the point C will be denoted by B and H + d, with $d \gtrless 0$ (the case with $d > 0$ is given in Fig. 1). The slit width is then $h = b\sin\lambda + d\cos\lambda$.

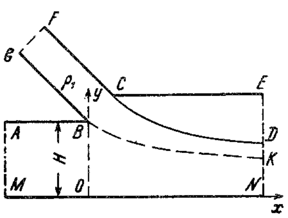

Fig. 1

We shall assume that on the streamline MON the stream function $\psi = 0$. If the gas flow rates at cross sections AM and FG are denoted by Q_2 and Q_1, respectively, and the gas flow rate at the cross section DN is denoted by Q,

$$Q = Q_1 + Q_2\,, \tag{1.1}$$

then the stream function $\psi = Q_2$ along the streamlines AB and GBK, which meet at the point B, and $\psi = Q$ on the stream line FCD.

In the hodograph velocity plane $\tau\theta$ (with polar coordinates $\tau = v^2/v^2_{max}$, where v is the velocity, v_{max} is the maximum velocity, and θ is the angle of inclination of the velocity to the x axis) the flow region under consideration is a circular sector of radius τ_3 and aperture angle λ (Fig. 2).

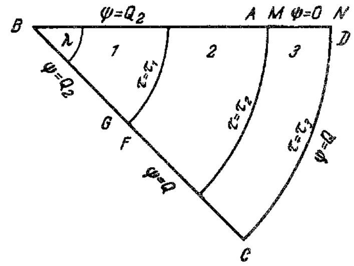

Fig. 2

The boundary conditions are

$$\psi = Q_2 \quad \text{for } \theta = -\lambda,\ 0 < \tau < \tau_1,$$

$$\psi = Q_2 \quad \text{for } \theta = 0,\ 0 < \tau < \tau_2\,,$$

$$\psi = Q \quad \text{for } \theta = -\lambda,\ \tau_1 < \tau < \tau_3,$$

$$\psi = 0 \quad \text{for } \theta = 0,\ \tau_2 < \tau < \tau_3\,, \tag{1.2}$$

$$\psi = Q \quad \text{for } -\lambda < \theta < 0,\ \tau = \tau_3\,. \tag{1.3}$$

Thus the solution of the present problem has been reduced to finding the solution of the internal Dirichlet problem for the Chaplygin equation

$$4\tau^2(1-\tau)\frac{\partial^2\psi}{\partial\tau^2} + 4\tau[1+(\beta-1)\tau]\frac{\partial\psi}{\partial\tau} + [1-(2\beta+1)\tau]\frac{\partial^2\psi}{\partial\theta^2} = 0\,,$$

$$\beta = 1/(\varkappa - 1), \quad \varkappa = c_p/c_v \tag{1.4}$$

in the appropriate regions of the circular sector.

Following Fal'kovich [2], we shall look for a solution of the problem in the form

$$\psi_1 = Q_2 + \sum_{n=1}^{\infty} a_n z_\omega(\tau)\sin 2\omega\theta \quad \left(\omega = \frac{n\pi}{2\lambda}\right), \tag{1.5}$$

$$\psi_2 = Q_2 - Q_1\frac{\theta}{\lambda} + \sum_{n=1}^{\infty}[A_n z_\omega(\tau) + B_n\zeta_\omega(\tau)]\sin 2\omega\theta\,, \tag{1.6}$$

$$\psi_3 = -Q\frac{\theta}{\lambda} + \sum_{n=1}^{\infty}[C_n z_\omega(\tau) + D_n\zeta_\omega(\tau)]\sin 2\omega\theta\,. \tag{1.7}$$

Here the ψ subscript corresponds to the number of that region of the circular sector for which we are seeking a solution; $z_\omega(\tau)$ is the solution of the equation

$$\tau^2(1-\tau)z_\omega'' + \tau[1+(\beta-1)\tau]z_\omega' - \omega^2[1-(2\beta+1)\tau]z_\omega = 0 \tag{1.8}$$

which is bounded at $\tau = 0$; $\zeta_\omega(\tau)$ is Cherry's function [4], a second linearly independent solution of Eq. (1.8), treated by Fal'kovich [2]. It is significant that the Wronskian of these integrals is

$$W(\tau) = \begin{vmatrix} z_\omega'(\tau) & \zeta_\omega'(\tau) \\ z_\omega(\tau) & \zeta_\omega(\tau) \end{vmatrix} = \frac{\omega}{\tau}(1-\tau)^\beta . \qquad (1.9)$$

The coefficients a_n, A_n, C_n, and D_n are to be determined.

The stream function specified by Eqs. (1.5)–(1.7), satisfies the boundary conditions (1.2). We now require that the boundary condition (1.3) be satisfied, and also that ψ_2 should be the analytic continuation of ψ_1 from region (1) to region (2), and that ψ_3 should be the analytic continuation of ψ_2 from region (2) to region (3), i.e., we require that for $-\lambda < \theta < 0$ the following equations should hold:

$$\psi_3(\tau_3, \theta) = Q, \quad \psi_1(\tau_1, \theta) = \psi_2(\tau_1, \theta),$$
$$\frac{\partial \psi_1(\tau_1, \theta)}{\partial \tau} = \frac{\partial \psi_2(\tau_1, \theta)}{\partial \tau},$$
$$\psi_2(\tau_2, \theta) = \psi_3(\tau_2, \theta), \quad \frac{\partial \psi_2(\tau_2, \theta)}{\partial \tau} = \frac{\partial \psi_3(\tau_2, \theta)}{\partial \tau} . \qquad (1.10)$$

Setting ψ_1, ψ_2, ψ_3 in (1.10) in accordance with (1.5)–(1.7) and equating the coefficients of $\sin 2\omega\theta$, we obtain the following system of equations for determining the coefficients:

$$C_n z_\omega(\tau_3) + D_n \zeta_\omega(\tau_3) = -2Q / n\pi,$$
$$(C_n - A_n) z_\omega(\tau_2) + (D_n - B_n)\zeta_\omega(\tau_2) = -2Q_2 / n\pi ,$$
$$(A_n - a_n) z_\omega(\tau_1) + B_n \zeta_\omega(\tau_1) = -(-1)^n 2Q_1 / n\pi ,$$
$$(A_n - a_n) z_\omega'(\tau_1) + B_n \zeta_\omega'(\tau_1) = 0,$$
$$(C_n - A_n) z_\omega'(\tau_2) + (D_n - B_n)\zeta_\omega'(\tau_2) = 0 . \qquad (1.11)$$

Solving the system of equations (1.11) and using the relation (1.9), we find the coefficients a_n, A_n, B_n, C_n, and D_n. The stream function ψ is found at the same time. In what follows we shall need only the function ψ in the region $\tau_2 < \tau < \tau_3$, i.e., ψ_3, which we shall denote simply by ψ. Inserting the coefficients C_n and D_n in (1.7), we have

$$\psi = \frac{Q}{\lambda}\Big[-\theta + \sum_{n=1}^{\infty} f_\omega(\tau) \frac{\sin 2\omega\theta}{\omega}\Big], \qquad (1.12)$$

$$f_\omega(\tau) = -\frac{z_\omega(\tau)}{z_\omega(\tau_3)} + \Big[\frac{\sigma_2 \tau_2}{\omega(1-\tau_2)^\beta} \frac{z_\omega'(\tau_2)}{z_\omega(\tau_3)} + (-1)^n \frac{\sigma_1 \tau_1}{\omega(1-\tau_1)^\beta} \frac{z'_\omega(\tau_1)}{z_\omega(\tau_3)}\Big] T_\omega(\tau, \tau_3),$$
$$T_\omega(\tau, \tau_3) = z_\omega(\tau)\zeta_\omega(\tau_3) - \zeta_\omega(\tau) z_\omega(\tau_3),$$
$$\sigma_1 = Q_1 / Q, \quad \sigma_2 = Q_2 / Q . \qquad (1.13)$$

We note that in what follows,

$$T_\omega'(\tau_i, \tau_3) = [T_\omega'(\tau, \tau_3)]_{\tau=\tau_i} \qquad (i = 1, 2, 3),$$
$$T_\omega'(\tau_i, \tau_i) = w(\tau_i), \quad T_\omega(\tau_i, \tau_i) = 0, \quad f_\omega(\tau_3) = -1,$$
$$f_\omega'(\tau_3) = -\frac{z_\omega'(\tau_3)}{z_\omega(\tau_3)} + \sigma_2 \frac{\tau_2}{\tau_3}\Big(\frac{1-\tau_3}{1-\tau_2}\Big)^\beta \frac{z_\omega'(\tau_2)}{z_\omega(\tau_3)} + (-1)^n \sigma_1 \frac{\tau_1}{\tau_3}\Big(\frac{1-\tau_3}{1-\tau_1}\Big)^\beta \frac{z_\omega'(\tau_1)}{z_\omega(\tau_3)} . \qquad (1.14)$$

§ 2. We shall determine the compression coefficient of the combined stream. Along the streamline the following general formula holds:

$$\frac{\partial y}{\partial \theta} = 2\tau \frac{(1-\tau)^{-\beta}}{v} \frac{\partial \psi}{\partial \tau} \sin\theta . \qquad (2.1)$$

Inserting the stream function ψ from (1.12) into (2.1), and setting $\tau = \tau_3$, we integrate from $-\lambda$ to θ. Keeping in mind that $y = H + d$ for $\theta = -\lambda$, we obtain the y ordinate along the jet CD,

$$y = \frac{Q}{\lambda} \frac{\tau_3(1-\tau_3)^{-\beta}}{v_3} \Big\{\sum_{n=1}^{\infty} \frac{f'_\omega(\tau_3)}{\omega}\Big[\frac{\sin(2\omega-1)\theta}{2\omega-1} - \frac{\sin(2\omega+1)\theta}{2\omega+1}\Big] - 4\sin\lambda \sum_{n=1}^{\infty} (-1)^n \frac{f_\omega'(\tau_3)}{4\omega^2-1}\Big\} + H + d . \qquad (2.2)$$

Remembering that the flow rate is $Q = \delta v_3(1-\tau_3)^\beta$ and that the condition $\theta = 0$, $y = \delta$ holds at infinity, then (2.2) may be easily transformed into the form

$$\delta = (H+d)\Big[1 + \frac{4\tau_3}{\lambda}\sin\lambda \sum_{n=1}^{\infty} (-1)^n \frac{f_\omega'(\tau_3)}{4\omega^2-1}\Big]^{-1} . \qquad (2.3)$$

The compression coefficient of the combined stream k will be taken to mean the ratio of the least width δ of the stream to the width $h + H$ of the slit and the channel. We then have directly from (2.3)

$$\frac{1}{k} = \frac{h+H}{H+d}\Big[1 + \frac{4\tau_3}{\lambda}\sin\lambda \sum_{n=1}^{\infty} (-1)^n \frac{f_\omega'(\tau_3)}{4\omega^2-1}\Big] \qquad \Big(k = \frac{\delta}{h+H}\Big). \qquad (2.4)$$

In addition to formulas (2.3) and (2.4) we require the equation of continuity (1.1), which may be written in the form

$$Q = hv_1(1-\tau_1)^\beta + Hv_2(1-\tau_2)^\beta = \delta v_3 (1-\tau_3)^\beta . \qquad (2.5)$$

Relations (2.3), (2.4), and (2.5) determine the stream velocity v_3, its width δ, and also the compression coefficient k in the functions v_1, v_2, h, H. In the particular case when $\lambda = \pi/2$, $d = 0$ and $\omega = n$ we have for the stream compression coefficient

$$\frac{1}{k} = \frac{h+H}{H}\Big\{1 - \frac{8\tau_3}{\pi}\Big[\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} \frac{z_n'(\tau_3)}{z_n(\tau_3)} - \sigma_2 \frac{\tau_2}{\tau_3}\Big(\frac{1-\tau_3}{1-\tau_2}\Big)^\beta \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} \frac{z_n'(\tau_2)}{z_n(\tau_3)} - \sigma_1 \frac{\tau_1}{\tau_3}\Big(\frac{1-\tau_3}{1-\tau_1}\Big)^\beta \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \frac{z_n'(\tau_1)}{z_n(\tau_2)}\Big]\Big\} \qquad (2.6)$$

when (1.14) is taken into account.

If the jet flows out of an orifice from an infinitely wide vessel, then $\tau_1 = 0$. For $\tau_2 = 0$, $H \to \infty$, and formula (2.6) passes to the well-known formula of Chaplygin [1]

$$\frac{1}{k_\infty} = 1 - \frac{8\tau_3}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} \frac{z_n'(\tau_3)}{z_n(\tau_3)} . \qquad (2.7)$$

Table 1

τ_3	τ_2	τ_1	h/H	k
0.04	0.02	0.02	0.0535	0.7274
0.06	0.02	0.02	0.1447	0.6119
	0.04	0.02	0.0240	0.8355
	0.04	0.04	0.0140	0.8409
0.08	0.02	0.02	0.2366	0.5459
	0.04	0.02	0.1642	0.7214
	0.04	0.04	0.1022	0.7502
	0.06	0.02	0.0128	0.8877
	0.06	0.04	0.0079	0.8910
	0.06	0.06	0.0056	0.8920
0.10	0.02	0.02	0.3208	0.5040
	0.04	0.02	0.1217	0.6722
	0.04	0.04	0.0782	0.6926
	0.06	0.02	0.0391	0.8115
	0.06	0.04	0.0247	0.8204
	0.06	0.06	0.0186	0.8235
0.12	0.02	0.02	0.3975	0.4744
	0.04	0.02	0.1718	0.6257
	0.04	0.04	0.1126	0.6518
	0.06	0.02	0.0689	0.7556
	0.06	0.04	0.0451	0.7697
	0.06	0.06	0.0342	0.7750

τ_3	τ_2	τ_1	h/H	k
0.12	0.08	0.02	0.0223	0.8600
	0.08	0.04	0.0146	0.8655
	0.08	0.06	0.0111	0.8675
	0.08	0.08	0.0089	0.8686
0.14	0.02	0.02	0.4664	0.4535
	0.04	0.02	0.2285	0.5902
	0.04	0.04	0.1448	0.6233
	0.08	0.02	0.0400	0.8163
	0.08	0.04	0.0265	0.8255
	0.08	0.06	0.0204	0.8290
	0.08	0.08	0.0167	0.8308
	0.10	0.02	0.0132	0.8941
	0.10	0.04	0.0087	0.8922
	0.10	0.06	0.0067	0.8939
	0.10	0.08	0.0055	0.8950
	0.10	0.10	0.0047	0.8999
1/6	0.02	0.02	0.5470	0.4347
	0.04	0.02	0.2761	0.5625
	0.04	0.04	0.1834	0.5977
	0.06	0.02	0.1357	0.6762
	0.06	0.04	0.0908	0.6997
	0.06	0.06	0.0708	0.7090
	0.08	0.02	0.0626	0.7719
	0.08	0.04	0.0419	0.7851
	0.08	0.06	0.0326	0.7903
	0.08	0.08	0.0271	0.7930

Table 2

τ_2/τ_3	τ_1/τ_3	h/H	k
0.50	0.50	0.0626	0.7071
0.3333	0.3333	0.1500	0.5773
0.6667	0.3333	0.0324	0.8088
0.6667	0.6667	0.0184	0.8166
0.2500	0.2500	0.2973	0.5000
0.5000	0.2500	0.1035	0.6877
0.7500	0.2500	0.0216	0.8584
0.7500	0.5000	0.0130	0.8641
0.7500	0.7500	0.0086	0.8662
0.2000	0.2000	0.4156	0.4472
0.4000	0.2000	0.1890	0.6018
0.4000	0.4000	0.1191	0.6310
0.6000	0.2000	0.0706	0.7531
0.6000	0.4000	0.0445	0.7686
0.6000	0.6000	0.0315	0.7747
0.8000	0.2000	0.0155	0.8876
0.8000	0.4000	0.0097	0.8919
0.8000	0.6000	0.0069	0.8936
0.8000	0.8000	0.0049	0.8945
0.1667	0.1667	0.5285	0.4083
0.3333	0.1667	0.2745	0.5408
0.3333	0.3333	0.1765	0.5773
0.5000	0.1667	0.1328	0.6720

τ_2/τ_3	τ_1/τ_3	h/H	k
0.5000	0.3333	0.0854	0.6968
0.6667	0.1667	0.0522	0.7963
0.6667	0.3333	0.0335	0.8088
0.6667	0.5000	0.0246	0.8140
0.1428	0.1428	0.6368	0.3779
0.2857	0.1428	0.3628	0.4929
0.2857	0.2857	0.2361	0.5346
0.5714	0.1428	0.1013	0.7211
0.5714	0.2857	0.0659	0.7321
0.5714	0.4286	0.0494	0.7510
0.5714	0.5714	0.0388	0.7558
0.7143	0.1428	0.0406	0.8270
0.7143	0.2857	0.0264	0.8373
0.7143	0.4286	0.0198	0.8416
0.7143	0.5714	0.0156	0.8439
0.7143	0.7143	0.0123	0.8453
0.12	0.12	0.7727	0.3464
0.24	0.12	0.4796	0.4534
0.24	0.24	0.3173	0.4899
0.36	0.12	0.2976	0.5418
0.36	0.24	0.1970	0.5818
0.36	0.36	0.1498	0.6000
0.48	0.12	0.1781	0.6396
0.48	0.24	0.1179	0.6707
0.48	0.36	0.0896	0.6848
0.48	0.48	0.0720	0.6920

Detailed calculations using this formula have been made in [3].

Taking into account (2.7) and (1.13), which may be reduced to

$$\sigma_1 = \frac{h}{\delta}\left(\frac{\tau_1}{\tau_3}\right)^{1/2}\left(\frac{1-\tau_1}{1-\tau_3}\right)^{\beta}, \qquad \sigma_2 = \frac{H}{\delta}\left(\frac{\tau_2}{\tau_3}\right)^{1/2}\left(\frac{1-\tau_2}{1-\tau_3}\right)^{\beta},$$

we may transform (2.5) and (2.6) to the following formulas which are convenient for making calculations:

$$k = \frac{\nu}{1+\nu}\left(\frac{\tau_1}{\tau_3}\right)^{1/2}\left(\frac{1-\tau_1}{1-\tau_3}\right)^{\beta} + \frac{1}{1+\nu}\left(\frac{\tau_2}{\tau_3}\right)^{1/2}\times$$
$$\times\left(\frac{1-\tau_2}{1-\tau_3}\right)^{\beta}\left(\nu = \frac{h}{H} = \frac{1-\Omega_2(\tau_2)}{\Omega_1(\tau_1)}\right),$$
$$\Omega_2(\tau_2) = \frac{1}{k_\infty}\left(\frac{\tau_2}{\tau_3}\right)^{1/2}\left(\frac{1-\tau_2}{1-\tau_3}\right)^{\beta} +$$
$$+\frac{8}{\pi}\left(\frac{\tau_2^3}{\tau_3}\right)^{1/2}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\,\frac{z_n'(\tau_2)}{z_n(\tau_3)},$$
$$\Omega_1(\tau_1) = \frac{1}{k_\infty}\left(\frac{\tau_1}{\tau_3}\right)^{1/2}\left(\frac{1-\tau_1}{1-\tau^3}\right)^{\beta} +$$
$$+\frac{8}{\pi}\left(\frac{\tau_1^3}{\tau_3}\right)^{1/2}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\,\frac{z_n'(\tau_1)}{z_n(\tau_3)}. \qquad (2.8)$$

In particular, for an incompressible fluid

$$\lim_{\tau_{i,j}\to 0}\frac{\tau_i z_n'(\tau_i)}{z_n(\tau_j)} = n\left(\frac{\tau_i}{\tau_j}\right)^n \qquad (i, j = 1, 2, 3)$$

and series (2.8) may be summed easily. As a result of summing the series in (2.8) we can obtain without difficulty

$$k = \frac{\nu}{1+\nu}\left(\frac{\tau_1}{\tau_3}\right)^{1/2} + \frac{1}{1+\nu}\left(\frac{\tau_2}{\tau^3}\right)^{1/2},$$

$$\frac{h}{H} = \left\{1 - \left[\left(\frac{\tau_2}{\tau_3}\right)^{1/2} + \frac{2}{\pi}\left(1-\frac{\tau_2}{\tau_3}\right)\operatorname{arc\,tg}\left(\frac{\tau_2}{\tau_3}\right)^{1/2}\right]\right\}\times$$
$$\times\left[\left(\frac{\tau_1}{\tau_3}\right)^{1/2} + \frac{2}{\pi}\left(1+\frac{\tau_1}{\tau_3}\right)\operatorname{Ar\,th}\left(\frac{\tau_1}{\tau_3}\right)^{1/2}\right]^{-1}. \qquad (2.9)$$

Calculations using (2.8) and (2.9) were carried out with an accuracy to four decimal places. The results of the calculations are given, respectively, in Tables 1 and 2. At the same time the ratios τ_2/τ_3 and τ_1/τ_3 for an incompressible fluid were calculated for the same values of τ_1, τ_2, and τ_3 as in the case of a compressible fluid. In making these calculations the values of the functions $z_n(\tau)$ and $z_n'(\tau)$ were taken from the tables of [3], and the results of the calculations of paper [5] were employed.

REFERENCES

1. S. A. Chaplygin, Gas Jets [in Russian], Gostekhizdat, 1949.

2. S. V. Fal'kovich, "The theory of gas jets," PMM, vol. 21, no. 4, 1957.

3. D. F. Ferguson and M. G. Lighthill, "The hodograph transformation in transonic flow, IV. Tables," Proc. Roy. Soc. A, vol. 192, no. 1028, 1947.

4. T. M. Cherry, "Asymptotic expansions for the hypergeometric functions occurring in gas-flow theory," Proc. Roy. Soc. London A, vol. 202, no. 1071, 1950.

5. S. K. Aslanov and V. A. Legkova, "The flow of a gas jet from a vessel of finite width," PMM, vol. 23, no. 1, 1959.

6 September 1966 Tomsk